MATH 1700: SECTION 11.2: MORE IDENTITIES

THE EVEN / ODD IDENTITIES: For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

THE SUM AND DIFFERENCE IDENTITIES: For all angles α and β,

- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

EXAMPLE 1:

1. Using the fact that $15^\circ = 45^\circ - 30^\circ$, find the exact value of $\cos(15^\circ)$.

2. Verify the identity: $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

3. Suppose α is a Quadrant I angle with $\sin(\alpha) = \frac{3}{5}$ and β is a Quadrant IV angle with $\sec(\beta) = 4$.

   Find the exact value of $\cos(\alpha + \beta)$.

4. Using the fact that $\frac{19\pi}{12} = \frac{4\pi}{3} + \frac{\pi}{4}$, find the exact value of $\sin\left(\frac{19\pi}{12}\right)$

5. Suppose α is a Quadrant II angle with $\sin(\alpha) = \frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta) = 2$.

   Find the exact value of $\sin(\alpha - \beta)$.

6. Derive a formula for $\tan(\alpha + \beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

The sum and difference formulas can be used to derive all of the following identities:

COFUNCTION IDENTITIES: For all applicable angles θ,

- $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$
- $\sec\left(\frac{\pi}{2}-\theta\right)=\csc(\theta)$
- $\tan\left(\frac{\pi}{2}-\theta\right)=\cot(\theta)$
- $\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$
- $\csc\left(\frac{\pi}{2}-\theta\right)=\sec(\theta)$
- $\cot\left(\frac{\pi}{2}-\theta\right)=\tan(\theta)$

SINUSOIDS, REVISITED:

Using the sum formulas for sine and cosine, we may expand sinusoids as:

$$S(t)=A\sin(\omega t+\phi)+B=A\sin(\omega t)\cos(\phi)+A\cos(\omega t)\sin(\phi)+B,$$

and

$$C(t)=A\cos(\omega t+\phi)+B=A\cos(\omega t)\cos(\phi)-A\sin(\omega t)\sin(\phi)+B.$$

EXAMPLE: Consider the function $f(t)=\cos(2t)-\sqrt{3}\sin(2t)$. Find a formula for $f(t)$:

1. in the form $C(t)=A\cos(\omega t+\phi)+B$ for $\omega>0$

2. in the form $S(t)=A\sin(\omega t+\phi)+B$ for $\omega>0$

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THE DOUBLE ANGLE IDENTITIES: For all applicable angles θ,

- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2(\theta)}$

EXAMPLE 2:

1. Suppose $P(-3, 4)$ lies on the terminal side of θ when θ is plotted in standard position.

   Find $\cos(2\theta)$ and $\sin(2\theta)$.

   In which quadrant does the terminal side of the angle 2θ lies when it is plotted in standard position?

2. If $\sin(\theta) = x$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

3. Verify the identity: $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.

4. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

We may rewrite the double angle identity for cosine to produce the following which prove useful in Calculus:

POWER REDUCTION FORMULAS: For all angles θ,

- $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$
- $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$

EXAMPLE 3: Rewrite $\sin^2(\theta)\cos^2(\theta)$ as a sum and difference of cosines to the first power.

Re-casting the roles of θ and 2θ in the power reduction identities gives us:

HALF ANGLE FORMULAS: For all applicable angles θ,

- $\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1+\cos(\theta)}{2}}$
- $\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos(\theta)}{2}}$
- $\tan\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\dfrac{\theta}{2}$ lies.

EXAMPLE 4:

1. Use a half angle formula to find the exact value of $\cos(15^\circ)$.

2. Suppose $-\pi \leq t \leq 0$ with $\cos(t) = -\frac{3}{5}$. Find $\sin\left(\frac{t}{2}\right)$.

3. Use the identity: $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$ to show: $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{\sin(\theta)}{1+\cos(\theta)}$

We may use the sum and difference formulas to derive the following:

PRODUCT TO SUM FORMULAS: For all angles α and β,

- $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$
- $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$
- $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right]$

Rewriting the above, we can also get the following:

SUM TO PRODUCT FORMULAS: For all angles α and β,

- $\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$
- $\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$
- $\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right)$

EXAMPLE 5:

1. Write $\cos(2\theta)\cos(6\theta)$ as a sum.

2. Write $\sin(\theta) - \sin(3\theta)$ as a product.

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